Self-Fourier functions

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LETTER TO THE EDITOR

Self-Fourier functions

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Abstract. The Gaussian and Dirac comb are often quoted as the only functions which are their own Fourier transform (FT), $\bar{f}(u)=f(u)$. We show that for arbitrary transformable $g(x)$, the function $f(x)=g(x)+g(-x)+\bar{g}(x)+\bar{g}(-x)$ is its own FT. We give physically reasonable examples and discuss some optical consequences.

Some functions $f(x)$ are their own Fourier transform (FT), i.e.,

$$\bar{f}(u)=f(u) \tag{1}$$

where

$$\bar{f}(u)=\int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{2\pi i u x}\,\mathrm{d}x \tag{2}$$

is the FT of $f(x)$. Two examples are a Gaussian

$$\begin{aligned} f(x)&=\mathrm{e}^{-\pi x^2}\\ \bar{f}(u)&=\mathrm{e}^{-\pi u^2} \end{aligned} \tag{3}$$

and a Dirac comb

$$\begin{aligned} f(x)&=\sum_n \delta(x-n)\\ \bar{f}(u)&=\sum_n \delta(u-n). \end{aligned} \tag{4}$$

Call them self-Fourier functions (SFFs). The literature gives the impression that (3) and (4) are perhaps the only ones [1]. We wish to point out that there is an infinity of SFFs: given an arbitrary transformable function $g(x)$, then the even function

$$f(x)=g(x)+g(-x)+\bar{g}(x)+\bar{g}(-x) \tag{5}$$

is an SFF, with FT

$$\bar{f}(u)=g(u)+g(-u)+\bar{g}(u)+\bar{g}(-u). \tag{6}$$

Equation (6) may be derived by taking the FT of the RHS of (5), using (2) and the delta-function

$$\delta(x_1-x_2)=\int \exp[2\pi i(x_1-x_2)x]\,\mathrm{d}x. \tag{7}$$

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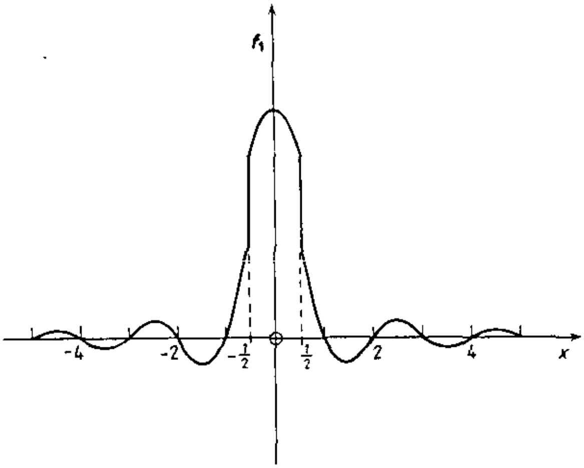

Figure 1. Self-Fourier function $f_1 = \text{rect}(x) + \text{sinc}(x)$.

In physical applications g and $\bar{g}$ are often different in nature (e.g. g is a signal and $\bar{g}$ its spectrum); thus combining g with $\bar{g}$ in (5) may seem unphysical and 'dimensionally' wrong. However, in (2) the variable u could as well be x', and x is a dummy variable which could be u': thus f of (5) is as physically reasonable a function as g and $\bar{g}$ separately are.

Explicitly,

$$f(x) = g(x) + g(-x) + 2\int_{-\infty}^{+\infty} g(x')\cos(2\pi x'x)\,dx' \tag{8}$$

is an SFF,

$$\bar{f}(u) = f(u). \tag{9}$$

Examples of SFFs are

$$f_1 = \text{rect}(x) + \text{sinc}(x) \tag{10}$$

$$f_2 = \Lambda(x) + \text{sinc}^2(x) \tag{11}$$

$$f_3 = e^{-|x|} + 2/(1 + 4\pi^2 x^2) \tag{12}$$

$$f_4 = 1 + \delta(x) \tag{13}$$

familiar in physical analysis. We sketch f_1 in figure 1, with $\text{rect}(x) = 1$ for $|x| \leqslant \frac{1}{2}$ and $= 0$ for $|x| > \frac{1}{2}$, and $\text{sinc}(x) = \sin(\pi x)/(\pi x)$; also, the triangle function $\Lambda(x) = 1 - |x|$ for $|x| \leqslant 1$ and $= 0$ for $|x| > 1$. Our one-dimensional analysis should be readily applicable to higher dimensions.

We briefly state two optical applications (see [1] for background). If we make a transparency whose amplitude transmissivity is (proportional to) $f(x)$ and use it as input to a coherent $2F$ Fourier processor, then the amplitude output is the FT, $\bar{f}(x/\lambda F)$. Thus if input $f(x)$ is an SFF, the output is a scaled *image* of the input; of the examples (10)–(13), f_2 is probably the easiest to realize optically. The second application is the design of laser resonator cavities.

Reference

[1] Lipson S G and Lipson H 1981 *Optical Physics* (Cambridge: Cambridge University Press) pp 190–1, 307–9. Many other texts state equations (3) and (4) but do not discuss the possibility of other SFFs.